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Critical, crossover and correction-to-scaling exponents for isotropic Lifshitz points to order $(8 - d)^2$

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Abstract

A two-loop renormalization group analysis of the critical behaviour at an isotropic Lifshitz point is presented. Using dimensional regularization and minimal subtraction of poles, we obtain the expansions of the critical exponents ν and η, the crossover exponent φ, as well as the (related) wave vector exponent β_q and the correction-to-scaling exponent ω to second order in $\epsilon_8 = 8 - d$. They are compared with the authors' recent ϵ-expansion results (2000 *Phys. Rev.* B **62** 12338, 2001 *Nucl. Phys.* B **612** 340) for the general case of an m-axial Lifshitz point. It is shown that the expansions obtained here by a direct calculation for the isotropic ($m = d$) Lifshitz point all follow from the latter upon setting $m = 8 - \epsilon_8$. This is so despite recent claims to the contrary by de Albuquerque and Leite (2002 *J. Phys. A: Math. Gen.* **35** 1807).

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1. Introduction

The concept of a Lifshitz point, introduced in 1975 [1–3], has attracted considerable attention during the past 25 years[3]. Recently, there has been renewed interest in the critical behaviour at such points. In particular, field-theory approaches have been utilized to determine the dimensionality expansions of various critical indices needed to characterize the critical behaviour at m-axial Lifshitz points [4–9].

We assume that the order-parameter symmetry which is spontaneously broken on the low-temperature side of the critical line on which the Lifshitz point is located is $O(n)$, and that the wave vector instability which sets in at this point is isotropic in the m-dimensional subspace of $\mathbb{R}^d$. A familiar model representing the universality class of d-dimensional systems

[3] For reviews and extensive lists of references, see [2–6]. The papers [4–6] contain lists of the most recent references on the subject.

with short-range interactions and an n-component order-parameter field $\phi(\boldsymbol{x})$ at such an (m, d, n)-Lifshitz point is described by the Hamiltonian

$$\mathcal{H} = \int \mathrm{d}^d x \left\{ \frac{\sigma_0}{2}(\triangle_\| \phi)^2 + \frac{1}{2}(\nabla_\perp \phi)^2 + \frac{\rho_0}{2}(\nabla_\| \phi)^2 + \frac{\tau_0}{2}\phi^2 + \frac{u_0}{4!}|\phi|^4 \right\}. \tag{1}$$

Here the position vector $\boldsymbol{x} = (\boldsymbol{x}_\|, \boldsymbol{x}_\perp)$ has an m-dimensional parallel component $\boldsymbol{x}_\|$ and a $(d-m)$-dimensional perpendicular component $\boldsymbol{x}_\perp$. In the Landau approximation, the Lifshitz point is located at $\tau_0 = \rho_0 = 0$; this approximation holds arbitrarily close to it for values of d exceeding the upper critical dimension, which is $d^*(m) = 4 + m/2$ for $m \leqslant 8$ [1].

Interest in the model (1) had already begun in 1975 [1, 10–12]. Unfortunately, early analyses [1, 10–12] based on Wilson's momentum shell recursion relations were rather incomplete and produced a long-standing controversy: the series expansions to second order in $\epsilon = d^*(m) - d$ that one group of authors derived for the correlation exponents η_{l2} and η_{l4} for general values of m [10] (or for $m = 1$ [11]) disagreed with those obtained by another group [12] for the special values $m = 2$ and $m = 6$. This state was certainly very unsatisfactory because the class of models defined via the Hamiltonian (1) is not only interesting in its own right, but also provides a natural, simple generalization of the standard class of n-component $|\phi|^4$-models (to which it reduces when $m = 0$). Moreover, besides representing universality classes, these models are of a prototypical nature in that they exhibit *anisotropic scale invariance*, offering excellent possibilities for investigating the question whether and under which conditions scale invariance might give rise to further invariances analogous to Schrödinger invariance or conformal symmetries [13, 14]. Clearly, thorough theoretical understanding of these models is highly desirable.

In two recent papers [4, 5], we have been able to perform a field-theoretic two-loop renormalization group (RG) analysis of the model (1) near its upper critical dimension $d^*(m)$ for general values of m with $0 < m < 8$. The results yield the ϵ expansions to second order of all critical, crossover, wave vector and correction-to-scaling exponents about any point $(m, d^*(m))$ on the critical line $d^*(m)$. The expansion coefficients involve four well-defined single integrals $j_\phi(m)$, $j_\sigma(m)$, $j_\rho(m)$ and $J_u(m)$, which we have been able to evaluate analytically for the special values $m = 0$, 2, 6, 8, and for other values of m by numerical integration. An appealing feature of these results is that *they include both isotropic cases, namely that of the usual critical point* ($m = 0$) *and that of the isotropic Lifshitz point* ($m = d = 8 - \epsilon_8$).

The requirement that our general m-dependent expressions reduce to the correct results for these simpler cases in which only the perpendicular or parallel parts of space remain provides crucial checks. These issues were briefly discussed in our previous work [5], where also a direct two-loop RG calculation for the isotropic case $m = d$ was announced but not described. Our aim here is to fill this gap by presenting details of such an analysis and a thorough discussion of the consistency of its results with our previous findings for general m.[4] We emphasize that this consistency is not *a priori* obvious because *both m* and *d* must be expanded about the value 8 while maintaining the equality $m = d$. As we shall explicitly verify, this consistency holds equally well on the levels of the final ϵ-expansion results, the counter-terms and the Feynman integrals. This demonstrates that the recent claim by de Albuquerque and Leite [16], who asserted that the critical exponents of the isotropic Lifshitz point cannot be obtained from those for general m, is unfounded.

[4] Such a calculation is analogous to that for the standard $|\phi|^4$ theory ($m = 0$), which can be found in many textbooks such as [15]. In order to address recent claims in [16], we believe that the explicit presentation of this calculation is necessary.

2. Two-loop results for the isotropic case $m = d$

In the isotropic case $m = d$, only the parallel part of space remains, so that $\boldsymbol{x}_{\|} = \boldsymbol{x}$. It is convenient to use the rescaled variables,

$$\phi(\boldsymbol{x}) = \sigma_0^{-d/8}\psi(\boldsymbol{y}) \qquad \boldsymbol{y} = \sigma_0^{-1/4}\boldsymbol{x}. \tag{2}$$

Then the Hamiltonian implied by equation (1) for $m = d$ becomes

$$\mathcal{H}^{(\mathrm{iso})} = \int \mathrm{d}^d y \left[\frac{1}{2}(\triangle\psi)^2 + \frac{\lambda_0}{2}(\nabla\psi)^2 + \frac{\tau_0}{2}\psi^2 + \frac{g_0}{4!}|\psi|^4\right] \tag{3}$$

with

$$g_0 = \sigma_0^{-d/4}u_0 \qquad \lambda_0 = \sigma_0^{-1/2}\rho_0. \tag{4}$$

We use dimensional regularization to regularize the ultraviolet (uv) singularities of the N-point cumulants $G^{(N)} = \langle\psi \ldots \psi\rangle^{\mathrm{cum}}$ and vertex functions $\Gamma^{(N)}$ of the theory in $d = 8 - \epsilon_8$ dimensions. These uv singularities can be absorbed via the reparametrizations

$$\psi = Z_\psi^{1/2}\psi_{\mathrm{ren}} \qquad g_0 F_d = \kappa^{\epsilon_8} Z_g g \tag{5}$$

and

$$\tau_0 - \tau_{\mathrm{LP}} = \kappa^4 Z_t t \qquad \lambda_0 - \lambda_{\mathrm{LP}} = \kappa^2 Z_\lambda \lambda \tag{6}$$

where τ_{LP} and λ_{LP}, the values of τ_0 and λ_0 at the Lifshitz point, vanish in our perturbative approach based on dimensional regularization. For the normalization constant in equation (5) we use the choice

$$F_d = \frac{2^{1-d}\pi^{-d/2}\Gamma\left(5 - \frac{d}{2}\right)\Gamma\left(\frac{d}{2} - 2\right)^2}{\Gamma(d-4)}. \tag{7}$$

In order to determine the renormalization factors Z_ψ, Z_g, Z_t and Z_λ, we compute the Laurent expansions of the vertex functions $\Gamma^{(2)}$, $\Gamma^{(4)}$, $\Gamma^{(2)}_{\psi^2}$ and $\Gamma^{(2)}_{(\nabla\psi)^2}$ at the Lifshitz point $t = \lambda = 0$, where the latter two involve insertions of (one half of) the indicated operators ψ^2 and $(\nabla\psi)^2$, respectively. The free propagator with which we work is given by

$$G(y) \equiv \int_{\boldsymbol{q}} \tilde{G}(\boldsymbol{q})\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}} = \frac{\pi^{-d/2}}{16}\Gamma[(d-4)/2]\,y^{4-d} \qquad \tilde{G}(\boldsymbol{q}) = \frac{1}{q^4} \tag{8}$$

in position and momentum space, respectively. The values of the one-loop integral

$$I_2(q) \equiv \int \mathrm{d}^d y\, G(y)^2\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}} \tag{9}$$

of $\tilde{\Gamma}^{(4)}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_4)$ (where $q = |\boldsymbol{q}_1 + \boldsymbol{q}_2|$ for example) and the two-loop integral

$$I_3(q) \equiv \int \mathrm{d}^d y\, G(y)^3\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}} \tag{10}$$

of $\tilde{\Gamma}^{(2)}(\boldsymbol{q})$ can be read from the Fourier transform of the generalized function $|\boldsymbol{y}|^\vartheta$, which is

$$\int \mathrm{d}^d y y^\vartheta\,\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}} = 2^{\vartheta+d}\pi^{d/2}\frac{\Gamma\left(\frac{\vartheta+d}{2}\right)}{\Gamma\left(-\frac{\vartheta}{2}\right)}q^{-\vartheta-d} \tag{11}$$

according to equation (2) of [17]. The results are listed in appendix A along with the other required two-loop integrals. We choose the renormalization factors such that the uv poles are minimally subtracted, obtaining

$$Z_\psi = 1 + \frac{n+2}{3}\frac{g^2}{160\epsilon_8} + \mathrm{O}(g^3) \tag{12}$$

$$Z_\psi^2 Z_g = 1 + \frac{n+8}{9}\frac{3}{2}\frac{g}{\epsilon_8} + \left[\left(\frac{n+8}{9}\frac{3}{2\epsilon_8}\right)^2 + 3\frac{5n+22}{27}\frac{1}{24\epsilon_8}\right] g^2 + \mathrm{O}(g^3) \quad (13)$$

$$Z_\psi Z_t = 1 + \frac{n+2}{6}\frac{g}{\epsilon_8} + \frac{n+2}{6}\left[\frac{n+5}{6}\frac{1}{\epsilon_8^2} + \frac{1}{24\epsilon_8}\right] g^2 + \mathrm{O}(g^3) \quad (14)$$

and

$$Z_\psi Z_\lambda = 1 + \frac{n+2}{3}\frac{3g^2}{32\epsilon_8} + \mathrm{O}(g^3). \quad (15)$$

From these results the beta and exponent functions, defined by

$$\beta_g(g) \equiv \kappa\,\partial_\kappa|_0\, g \quad (16)$$

and

$$\eta_\wp \equiv \kappa\,\partial_\kappa|_0 \ln Z_\wp \qquad \wp = \psi, t, \lambda \quad (17)$$

where $\partial_\kappa|_0$ denotes a derivative at fixed values of the bare variables λ_0, τ_0 and g_0, follow in a straightforward fashion. They read

$$\beta_g(g) = -\epsilon_8 g + \frac{n+8}{6} g^2 + \frac{41n+202}{1080} g^3 + \mathrm{O}(g^4) \quad (18)$$

$$\eta_\psi(g) = -\frac{n+2}{3}\frac{g^2}{80} + \mathrm{O}(g^3) \quad (19)$$

$$\eta_t(g) = \frac{n+2}{6} g - \frac{7(n+2)}{720} g^2 + \mathrm{O}(g^3) \quad (20)$$

and

$$\eta_\lambda(g) = -\frac{7(n+2)g^2}{120} + \mathrm{O}(g^3). \quad (21)$$

The nontrivial root of $\beta_g(g)$ is

$$g^* = \frac{6\epsilon_8}{n+8} - \frac{41n+202}{5(n+8)^3}\epsilon_8^2 + \mathrm{O}\left(\epsilon_8^3\right). \quad (22)$$

By evaluating the exponent functions at the fixed point g^*, we arrive at the dimensionality expansions of the exponents,

$$\eta = \eta_\psi(g^*) = -\frac{3(n+2)\epsilon_8^2}{20(n+8)^2} + \mathrm{O}\left(\epsilon_8^3\right) \quad (23)$$

$$\nu = \frac{1}{4+\eta_t(g^*)} = \frac{1}{4} + \frac{n+2}{n+8}\frac{\epsilon_8}{16} + \frac{(n+2)(15n^2+89n+4)\epsilon_8^2}{960(n+8)^3} + \mathrm{O}\left(\epsilon_8^3\right) \quad (24)$$

$$\varphi = \frac{2+\eta_\lambda(g^*)}{4+\eta_t(g^*)} = \frac{1}{2} + \frac{n+2}{n+8}\frac{\epsilon_8}{8} + \frac{(n+2)(15n^2-163n-2\,012)\epsilon_8^2}{480(n+8)^3} + \mathrm{O}\left(\epsilon_8^3\right) \quad (25)$$

and[5]

$$\beta_q = \frac{1}{2} + \frac{21(n+2)\epsilon_8^2}{40(n+8)^2} + \mathrm{O}\left(\epsilon_8^3\right). \quad (26)$$

[5] This result for β_q follows via the scaling law $\beta_q = \varphi/\nu_{l4}$. Strictly speaking, the conventional definition of the wave vector exponent β_q relies on the existence of a modulated ordered phase. For models having the presumed isotropy of the Hamiltonian (1) in the m-dimensional parallel subspace, one can argue that a long-range ordered helical phase should be absent when $m \leqslant d \leqslant m+1$ [6, 18, 19]. This includes the isotropic case.

Likewise we find for the correction-to-scaling exponent

$$\omega = \left.\frac{\partial\beta_g(g)}{\partial g}\right|_{g=g^*} = \epsilon_8 + \frac{(41n+202)\epsilon_8^2}{30(n+8)^2} + \mathrm{O}\left(\epsilon_8^3\right). \tag{27}$$

The results (23) and (24) were already obtained in [1][6], but not the remaining ones (25)–(27).

3. Comparison with the results for the m-axial Lifshitz point

3.1. Critical, crossover and correction-to-scaling exponents

In [5], which hereafter is referred to as I, we considered the critical exponents as functions $f(m,d)$ of m and d, and determined their expansions in d about $d^*(m)$ at *fixed m* to second order in $\epsilon = d^*(m) - d$. For $d = m = 8 - \epsilon_8$, we have

$$\epsilon = \epsilon_8/2. \tag{28}$$

The series expansion coefficients of the terms of zeroth and first order in ϵ, found in I, are independent of m. To check the consistency of the above results with those of [5, 9], we therefore must merely substitute the limiting values of the latter's $\mathrm{O}(\epsilon^2)$ terms for $m \to 8-$, and use relation (28).

The exponents η, ν and ω given in equations (23), (24) and (27) read in the notation of I

$$\eta = \eta_{l4}(d,d) \qquad \nu = \nu_{l4}(d,d) \qquad \omega = \omega_{l4}(d,d) \tag{29}$$

respectively. The ϵ expansions of η_{l4} and ν_{l4} are given in equations (I.62) and (I.64), and the one for ω_{l4} follows from equations (I.66)–(I.68), where (I.*xx*) denotes equation (*xx*) of I. Note that the contribution $\propto \lim_{m\to 8-} j_\phi(m)/(8-m) = -j'_\phi(8-)$ which is present in the ϵ expansion (I.66) of the anisotropy exponent θ, does not contribute to the ratio $\omega_{l4} = \omega_{l2}/\theta$ at $\mathrm{O}(\epsilon^2)$. The only term involving $j_\phi(m)$ at $\mathrm{O}(\epsilon^2)$ is directly proportional to $j_\phi(m)$. Since the latter integral is of order ϵ_8, all contributions to the series expansion of the correction-to-scaling exponent ω to $\mathrm{O}(\epsilon_8^2)$ that originate from $j_\phi(m)$ vanish, as they should.

Upon substituting the values (I.86) of the integrals $j_\phi(8-), \ldots, J_u(8-)$ into the $\mathrm{O}(\epsilon^2)$ coefficients, the resulting series expansions to order ϵ_8^2 of the exponents (29) reduce to the above results (23), (24) and (27). Likewise, the ϵ_8 expansions of $\varphi = (1+\eta_\rho^*)/(2+\eta_\tau^*)$ and $\beta_q = \nu_{l4}/\varphi$ that are implied by equations (I.63)–(I.65) for the fixed-point quantities η_τ^*, η_σ^* and η_ρ^* of I agree with equations (25) and (26). The remaining critical exponents, such as the specific-heat exponent α, the order-parameter exponent β and the susceptibility exponent γ, are related to η and ν via known scaling and hyperscaling relations. Thus, there is no need to explicitly verify their consistency with the results of I.

For the case of an m-axial Lifshitz point a variety of other physically meaningful critical exponents can be defined, such as the exponent η_{l2} (which governs the momentum dependence of the inverse correlation function $\tilde{\Gamma}^{(2)}(\boldsymbol{q}_\parallel = \boldsymbol{0}, \boldsymbol{q}_\perp) \sim q_\perp^{2-\eta_{l2}}$ at the Lifshitz point), the parallel correlation-length exponent ν_{l2}, the related anisotropy exponent $\theta = \nu_{l4}/\nu_{l2}$ and the correction-to-scaling exponent $\omega_{l2} = \theta\omega_{l4}$. The ϵ expansions to order ϵ^2 of all these exponents have been given in I. These latter exponents are not needed in the case of the isotropic Lifshitz point. The situation is complementary to the $m = 0$ case of the standard $|\phi|^4$ theory, where exponents requiring the parallel part of space (such as η_{l4} and ν_{l4}) lose their physical significance.

[6] When referring to the results of the work [1] at the bottom of page 355 of our previous paper [5], we erroneously stated that these authors had obtained the expansions of the correlation exponents ν_{l2} and η_{l4} and the correlation-length exponents ν_{l2} and ν_{l4} to second order in ϵ_8. The alert reader must have noted that the cited computation was concerned merely with the isotropic Lifshitz point, so that only $\eta_{l4} = \eta$ and $\nu_{l4} = \nu$ should have been mentioned.

Remarks analogous to those made in I apply here: the limits $d - m \to 0$ and $m \to 0$ of those exponents that are not required in the isotropic cases $m = d$ and $m = 0$ may well exist, and it is conceivable that their limiting values will turn out to have significance for certain problems of statistical physics. We will not pursue this question further here. Note, however, that the limiting value of η_{l2} for $m \to d$ appears to exist and to be finite since its ϵ^2 term involves the integral $j_\phi(m)$ in the combination $j_\phi(m)/(8 - m)$, according to equation (I.61). Since $j_\phi(8-) = 0$, the limit $m \to 8-$ of this combination is nonsingular and nonzero provided that $j'_\phi(8-) \neq 0$ (as we expect). For the same reason, the contribution $\propto j_\phi(m)/(8 - m)$ to ν_{l2} in equation (I.63) remains finite in this limit, as do the other contributions to the $\mathrm{O}(\epsilon^2)$ term.

3.2. Comparison of renormalization functions

In the previous section, we have shown that the correct series expansions in powers of ϵ_8 of the critical, crossover and correction-to-scaling exponents of the isotropic Lifshitz point follow from the results of I. An analogous result holds for other universal quantities. This is because the required renormalization factors Z_ψ, Z_g, Z_t and Z_λ, and hence the implied beta and exponent functions $\beta_g(g)$ and η_ψ, η_t, η_λ may all be obtained from the m-dependent results of I.

A direct way of looking at this is to compare the Feynman integrals on which the analysis of the present paper is based (and which are computed in appendix A) with their analogues of I. To facilitate this comparison, it is advisable to get rid of the choice of the factors F_d and $F_{m,\epsilon}$ that we absorbed in g and u (the renormalized coupling constant of I), respectively. To achieve this goal, we must merely divide two-loop Feynman integrals such as I_3, I_4 and I_5 by the square of $\epsilon_8 I_2$ (as can be seen from equations (A.1) and (I.38)); that is, the normalized integral $I_4(\boldsymbol{e}, \boldsymbol{0})/[\epsilon_8 I_2(\boldsymbol{e})]^2$ should be compared with the quantity $\{I_4(\boldsymbol{e}_\perp, \boldsymbol{0})/[\epsilon I_2(\boldsymbol{e}_\perp)]^2\}_{m=8-\epsilon_8,\epsilon=\epsilon_8/2}$, where $\boldsymbol{e}$ and $\boldsymbol{e}_\perp$ are unit d and $d - m$ vectors, respectively. Utilizing equations (I.B.14), (I.86) and (I.B.13) of I, and the corresponding results (A.1) and (A.8) for the isotropic case, one finds that the pole parts of these quantities agree indeed:

$$\left.\frac{I_4(\boldsymbol{e}_\perp, \boldsymbol{0})}{[I_2(\boldsymbol{e}_\perp)]^2}\right|_{m=8-\epsilon_8,\epsilon=\epsilon_8/2} = \frac{1}{2\epsilon_8}\left[\frac{1}{\epsilon_8} - \frac{1}{12} + \mathrm{O}(\epsilon_8)\right] = \frac{I_4(\boldsymbol{e}, \boldsymbol{0})}{[\epsilon I_2(\boldsymbol{e})]^2}. \tag{30}$$

Analogous results hold for the other integrals needed to determine the two-loop counter-terms. We have

$$\frac{I_3(\boldsymbol{q})}{[\epsilon_8 I_2(\boldsymbol{e})]^2} = \frac{3q^4}{80\epsilon_8} + \mathrm{O}\left(\epsilon_8^0\right) = \left\{\left[\frac{F_{m,\epsilon}^2}{\epsilon}\,\frac{j_\sigma(m)q^4}{16m(m+2)} + \mathrm{O}(\epsilon^0)\right]\frac{1}{[\epsilon I_2(\boldsymbol{e}_\perp)]^2}\right\}_{m=8-\epsilon_8,\epsilon=\epsilon_8/2} \tag{31}$$

and

$$\frac{I_5(\boldsymbol{q})}{[\epsilon_8 I_2(\boldsymbol{e})]^2} = \frac{-3q^2}{16\epsilon_8} + \mathrm{O}\left(\epsilon_8^0\right) = \left\{\left[\frac{-F_{m,\epsilon}^2}{\epsilon}\,\frac{j_\rho(m)}{4m}q^2 + \mathrm{O}(\epsilon^0)\right]\frac{1}{[\epsilon I_2(\boldsymbol{e}_\perp)]^2}\right\}_{m=8-\epsilon_8,\epsilon=\epsilon_8/2} \tag{32}$$

where $F_{m,\epsilon} = \epsilon I_2(\boldsymbol{e}_\perp)$ is the counterpart of the normalization constant F_d. The analogues of I_3 and I_5 we inserted on the right-hand sides of these equations can be read from equations (I.B.4) and (I.B.6). Thus the pole terms found in I reduce in the isotropic case precisely to those obtained here. Provided we use the same conventions as here to fix the counter-terms, we get identical results for the Z factors Z_ψ, Z_g, Z_t and Z_λ.

Let us briefly comment on the fact that we evaluated the integrals I_2 and I_4 in I at a momentum $\boldsymbol{q}$ with vanishing parallel component $\boldsymbol{q}_\parallel$ and finite perpendicular component $\boldsymbol{q}_\perp$.

In the limit $m \to d$, the latter becomes zero-dimensional. However, this is no cause for concern. First of all, we should get reasonable results in this limit (as our results in I show) because the opposite would indicate a failure of the analytic continuation of the momentum-space integrals $\int \mathrm{d}^{d-m}q$ in $d-m$. Second, taking $\boldsymbol{q} = (\boldsymbol{0}, \boldsymbol{q}_\perp)$ may be viewed as the special choice $\exp(\mathrm{i}\boldsymbol{q}_\perp \cdot \boldsymbol{x}_\perp - ax^2)$, with $a \to 0+$, of a test function utilized to determine the pole term of the corresponding distributions $G(\boldsymbol{x})^2$ and the Fourier back-transform of I_4 to position space. The result $G(\boldsymbol{x})^2 F_{m,\epsilon}^{-2} = \epsilon^{-1}\delta(\boldsymbol{x}) + \mathrm{O}(\epsilon^0)$ we obtained in I can equally well be derived by computing the action of G^2 on other test functions (integrating over a finite sphere centred at the origin would do). If we set $d = m = 8 - \epsilon_8$ in this result, we recover the correct pole term of the isotropic case. Needless to say, only the $(m = d)$-dimensional part $\delta(\boldsymbol{x}_\|)$ of the delta function remains for $d = m$, where $\boldsymbol{x} = \boldsymbol{x}_\|$.

By comparing the renormalized action implied by our reparametrizations (5) and (6) with the renormalized action of I when $m = d$, one can easily see how our two-loop results (12)–(15) for these renormalization factors can be expressed in terms of those of I. Since to order u^2 only the terms $\propto u^2/\epsilon$ depend on m, we can set $m = 8$ when evaluating them. The desired relations then become

$$Z_\psi(g, \epsilon_8) = [Z_\sigma Z_\phi]_{u=g/2;m=8,\epsilon=\epsilon_8/2} + \mathrm{O}(g^3) \tag{33}$$

$$\big(Z_\psi^2 Z_g\big)(g, \epsilon_8) = \big[Z_u Z_\phi^2 Z_\sigma^{m/4}\big]_{u=g/2;m=8,\epsilon=\epsilon_8/2} + \mathrm{O}(g^3) \tag{34}$$

$$(Z_\psi Z_t)(g, \epsilon_8) = [Z_\phi Z_\tau]_{u=g/2;m=8,\epsilon=\epsilon_8/2} + \mathrm{O}(g^3) \tag{35}$$

and

$$(Z_\psi Z_\lambda)(g, \epsilon_8) = \big[Z_\rho Z_\phi Z_\sigma^{1/2}\big]_{u=g/2;m=8,\epsilon=\epsilon_8/2} + \mathrm{O}(g^3) \tag{36}$$

where the replacement $u \to g/2$ is due to the different normalization constants whose ratio is $F_{m=8-\epsilon_8,\epsilon_8/2}/F_d = 1/2 + \mathrm{O}(\epsilon_8)$.

For the associated RG functions, this translates into

$$\beta_g(g; \epsilon_8) = -\epsilon_8 g + 4\beta_u(g/2; m = 8, \epsilon = 0) + \mathrm{O}(g^3) \tag{37}$$

$$\eta_\psi(g) = 2\eta_\sigma(g/2; m = 8) + \mathrm{O}(g^3) \qquad \eta_t(g) = 2\eta_\tau(g/2; 8) + \mathrm{O}(g^3) \tag{38}$$

and

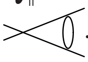
$$\eta_\lambda(g) = 2\eta_\rho(g/2; 8) - \eta_\sigma(g/2; 8) + \mathrm{O}(g^3). \tag{39}$$

The reader is invited to convince himself that the results (12)–(15) for the Z factors and the RG functions (18)–(21) of the isotropic Lifshitz point are indeed recovered from the above formulae when the m-dependent results of I are inserted into them, utilizing the required values (I.86) of the integrals $j_\phi(m), \ldots, J_u(m)$ at $m = 8$.

4. Concluding remarks

In summary, we have shown that the counter-terms, RG functions and the series expansions in powers of $\epsilon_8 = 8 - d$ of the critical, crossover and correction-to-scaling exponents of the isotropic Lifshitz point can be obtained from the results of I for the more general case of an m-axial Lifshitz point. We have verified their validity by performing an independent two-loop calculation directly for the isotropic case $d - m = 0$, using well-established standard field-theory techniques.

The fact that a proper field-theoretic analysis of the critical behaviour at m-axial Lifshitz points, based on the ϵ expansion, covers also the isotropic case $m = d$, should not be too surprising; after all, the free propagator $G(\boldsymbol{x})$ for general $m \neq d$ goes smoothly over into the correct isotropic propagator of equation (8) as $d - m \to 0$. The Feynman graphs of the primitively divergent vertex functions are distributions constructed from $G(\boldsymbol{x})$ having well-defined Laurent expansions in ϵ. The renormalization factors are determined by the pole parts of the final subtractions which the primitively divergent vertex functions require. Clearly, their behaviour in the isotropic limit should comply with that of $G(\boldsymbol{x})$.

In closing, let us straighten out another critique by de Abuquerque and Leite [16]. They claimed that our two-loop calculation was incomplete because we set the parallel component $\boldsymbol{Q}_\parallel$ of the momentum $\boldsymbol{Q}$ to zero when computing the two-loop integral $I_4(\boldsymbol{Q}, \boldsymbol{K})$ of the graph . However, the *final* subtractions we determined in this part of our computation were those associated with the $|\phi|^4$ counter-term and the vertex function $\Gamma^{(2)}_{\phi^2}$, both of which are *momentum independent*. It is true that $I_4(\boldsymbol{Q}, \boldsymbol{K})$ has first-order poles in ϵ that depend also on $\boldsymbol{Q}_\parallel$. They are induced by the divergent sub-integral , and are cancelled automatically by the subtractions provided by the one-loop counter-terms, in renormalizable theories such as those considered in I and here. Thus the convenient choice $\boldsymbol{Q}_\parallel = 0$ is possible. Readers who would like to see this reiterated and verified explicitly in the present context may consult appendix B.

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Appendix A. Feynman integrals

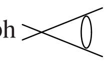
Equation (11) yields for the integrals (9) and (10) the results

$$I_2(q) = 2^{-d}\pi^{-\frac{d}{2}} \frac{q^{d-8}\Gamma\left(4-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-2\right)^2}{\Gamma(d-4)} = F_d \frac{q^{-\epsilon_8}}{\epsilon_8} \tag{A.1}$$

and

$$I_3(q) = F_d^2 \frac{q^{2(d-6)}\Gamma(6-d)\Gamma(d-4)^2}{4\Gamma\left(5-\frac{d}{2}\right)^2\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(\frac{3d}{2}-6\right)} \tag{A.2}$$

$$= F_d^2 q^4 \left[\frac{3}{80\epsilon_8} + \frac{211 - 240\ln q}{3200} + \mathrm{O}(\epsilon_8)\right] \tag{A.3}$$

respectively. The two-loop graph  involves the Feynman integral

$$I_4(\boldsymbol{q}_{12}, \boldsymbol{q}_3) \equiv \int_q \frac{1}{q^4|\boldsymbol{q} - \boldsymbol{q}_{12}|^4} I_2(|\boldsymbol{q} + \boldsymbol{q}_3|). \tag{A.4}$$

Using a familiar trick due to Feynman, we can transform the product of denominators into a single power by the introduction of two Feynman parameters s and t. Performing the integral over $\boldsymbol{q}$ then gives

$$I_4(\boldsymbol{q}_{12}, \boldsymbol{q}_3) = \frac{F_d}{\epsilon_8} \frac{\Gamma(\epsilon_8)}{(4\pi)^{d/2}\Gamma\left(\frac{\epsilon_8}{2}\right)} \int_0^1 \mathrm{d}s\, s(1-s) \int_0^1 \mathrm{d}t\, t^3(1-t)^{-1+\epsilon_8/2}$$
$$\times \left\{st q_{12}^2 + (1-t)q_3^2 - [(1-t)q_3 - st q_{12}]^2\right\}^{-\epsilon_8}. \tag{A.5}$$

By adding and subtracting the value of the integrand at $q_3 = 0$, one can easily see that the pole part of $I_4(\boldsymbol{q}_{12}, \boldsymbol{q}_3)$ is independent of $\boldsymbol{q}_3$ (noting that the pre-factor of the integral on the right-hand side is of order ϵ_8^{-1}). We therefore set $q_3 = 0$, which leads us to the integral

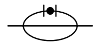
$$J(d) \equiv \int_0^1 \mathrm{d}s\, s^{1-\epsilon_8}(1-s) \int_0^1 \mathrm{d}t\, t^{3-\epsilon_8}(1-t)^{-1+\epsilon_8/2}(1-st)^{-\epsilon_8} \tag{A.6}$$

whose dimensionality expansion

$$J(8-\epsilon_8) = \frac{1}{3\epsilon_8} + \frac{1}{4} + \mathrm{O}(\epsilon_8) \tag{A.7}$$

can be determined in a straightforward fashion. Upon expanding the pre-factor of the integral, one arrives at the result

$$I_4(\boldsymbol{q}_{12}, \boldsymbol{q}_3) = F_d^2 \frac{q_{12}^{-2\epsilon_8}}{2\epsilon_8}\left[\frac{1}{\epsilon_8} - \frac{1}{12} + \mathrm{O}(\epsilon_8)\right]. \tag{A.8}$$

As usual, the required two-loop diagrams of $\Gamma^{(2)}_{\phi^2}$ involve just the integrals I_2 and I_4. However, we also need the integral associated with the two-loop graph  of $\tilde{\Gamma}^{(2)}_{(\nabla\psi)^2}(\boldsymbol{q}, \boldsymbol{Q})$, where $\boldsymbol{Q}$ denotes the momentum of the inserted operator $[(\nabla\psi)^2/2]_{\boldsymbol{Q}}$. Since the limit $\boldsymbol{Q} \to \boldsymbol{0}$ of this Feynman integral is not infrared-singular and exists, we can consider directly the case $\boldsymbol{Q} = \boldsymbol{0}$. The upper line with such an insertion corresponds in position space to the function

$$D(y) = \int_{\boldsymbol{q}} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}}}{q^6} = \frac{\pi^{-d/2}}{128}\Gamma\left(\frac{d-6}{2}\right) y^{6-d}. \tag{A.9}$$

Thus the required Feyman integral can be written as

$$I_5(q) = \int \mathrm{d}^d y\, G(y)^2 D(y)\, \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}}. \tag{A.10}$$

It can be evaluated in a straightforward fashion to obtain

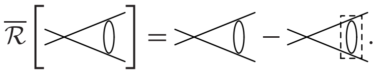
$$\begin{aligned} I_5(q)/F_d^2 &= \frac{\Gamma(7-d)\Gamma[(d-6)/2]\Gamma(d-4)^2}{8\Gamma[(10-d)/2]^2\,\Gamma[(d-4)/2]^2\,\Gamma[(3d/2)-7]} q^{2(d-7)} \\ &= -\frac{3q^2}{16\epsilon_8} + \mathrm{O}\left(\epsilon_8^0\right). \end{aligned} \tag{A.11}$$

Utilizing the Laurent expansions of the integrals given above, one can derive the results (12)–(15) for the renormalization factors in a straightforward manner.

Appendix B. Cancellation of momentum-dependent pole terms

In this appendix, we show explicitly for the case of the m-axial Lifshitz point that the momentum-dependent poles of the two-loop graph of $\Gamma^{(4)}$ appearing first on the right-hand side of equation (B.1) are cancelled by the subtractions which the one-loop counter-terms provide for the divergent sub-integral inside the dashed box. It is sufficient to consider the problem for a one-component order parameter ($n = 1$). The lines correspond to the free propagator of the Hamiltonian (1) for $\tau_0 = \rho_0 = 0$; its explicit forms in momentum and position space can be found in equations (I.5)–(I.11).

Zimmermann's forest formula [20] tells us, for any Feynman graph, which subtractions for its divergent subgraphs the lower-order counter-terms translate into. In our case, the result is particularly simple: the one-loop $|\phi|^4$ counter-term yields the subtraction



(B.1)

Here the notation $\overline{\mathcal{R}}[\gamma]$ is used to indicate the quantity that results from a Feynman integral $I[\gamma]$ of a graph γ by making the required subtractions for all its divergent subgraphs but not the final subtraction for γ itself. The boxed subgraph denotes its singular part. This is local in position space (namely, proportional to ϵ^{-1} times the $|\phi|^4$ vertex). Thus the subtracted graph involves (besides a momentum-independent second-order pole) a momentum-dependent first-order pole of the form ϵ^{-1} times the momentum-dependent term of order ϵ^0 of the graph that results when the box is contracted to a point. The latter is again the one-loop graph of $\Gamma^{(4)}$.

The pole terms of the difference (B.1) must be momentum independent and local in position space so that they can be absorbed via the $|\phi|^4$ counter-term. From equation (I.38) we know that the singular part of the divergent sub-integral I_2 of I_4 agrees with the pole part of $I_2(\boldsymbol{e}_\perp) = F_{m,\epsilon}\epsilon^{-1}$, where $F_{m,\epsilon}$ is the constant (I.39) while $\boldsymbol{e}_\perp$ denotes a unit $d-m$ vector. Hence the difference of Feynman integrals we are concerned with becomes

$$\overline{\mathcal{R}}[I_4](\boldsymbol{Q},\boldsymbol{K}) = \int_{\boldsymbol{q}} \frac{1}{q_\perp^2 + q_\parallel^4} \frac{1}{|\boldsymbol{q}_\perp + \boldsymbol{Q}_\perp|^2 + |\boldsymbol{q}_\parallel + \boldsymbol{Q}_\parallel|^4} [I_2(\boldsymbol{q}-\boldsymbol{K}) - I_2(\boldsymbol{e}_\perp)]. \tag{B.2}$$

We must show that the difference $\overline{\mathcal{R}}[I_4](\boldsymbol{Q},\boldsymbol{K}) - \overline{\mathcal{R}}[I_4](\boldsymbol{Q},\boldsymbol{K})|_{\boldsymbol{Q}_\parallel=\boldsymbol{0}}$ is regular in ϵ. This difference is given by the analogue of equation (B.2) that results through the replacement

$$\frac{1}{|\boldsymbol{q}_\perp + \boldsymbol{Q}_\perp|^2 + |\boldsymbol{q}_\parallel + \boldsymbol{Q}_\parallel|^4} \to \frac{-Q_\parallel^2(Q_\parallel^2 + 2q_\parallel^2) - 4(\boldsymbol{Q}_\parallel\cdot\boldsymbol{q}_\parallel + q_\parallel^2 + Q_\parallel^2)\boldsymbol{Q}_\parallel\cdot\boldsymbol{q}_\parallel}{[|\boldsymbol{q}_\perp + \boldsymbol{Q}_\perp|^2 + |\boldsymbol{q}_\parallel + \boldsymbol{Q}_\parallel|^4]\,[|\boldsymbol{q}_\perp + \boldsymbol{Q}_\perp|^2 + q_\parallel^4]}. \tag{B.3}$$

The subtraction has improved the uv behaviour of the integral by two powers of $q_\parallel$, making it uv convergent at the upper critical dimension in accordance with Weinberg's theorem [21]. Hence the momentum-dependent poles—and especially the $\boldsymbol{Q}_\parallel$-dependent ones—cancel, as they should, both in $\Gamma^{(4)}$ as well as in $\Gamma^{(2)}_{\phi^2}$.

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